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A NEW FINITE ELEMENT APPROACH TO HEAT FLOW ANALYSIS IN 3D DEVELOPABLE STRUCTURES

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A finite element approach to the thermal analysis of a thin structure located in three-dimensional space is presented. The problem is reduced to a two-dimensional one with special boundary conditions. A comparison of the results obtained in the paper with the full 3D solution is discussed.

Three-dimensional finite elements are usually used to analyse three dimensional field problems. However, in some cases it is convenient to replace three-dimensional thin structures by two-dimensional ones. Since descriptive geometry presents spatial problems as plane problems, this seems to be the most suitable tool for such an analysis. It is well known that every three-dimensional thin structure may be shown as a two-dimensional one through an appropriate developed view. We shall use specific boundary conditions in developing the structure.

Developed view of the three-dimensional structure

Suppose that the three-dimensional thin surface Ω^3 is divided into finite elements, and the region Ω^2_D is the two-dimensional developed view of Ω^3, obtained in such a way that cutting lines lie along the contact lines of finite elements. For instance, for the surface of the cylinder Ω^3 with discretization meshes shown in Fig. 1, the developed view of this cylinder may be shown as in Fig. 2. Another example showing a box is given in Figs 3 and 4. It is well known that for every three-dimensional surface we can find an appropriate developed view by using approximations in some cases. The methods of finding the developed view are given in different text-books on geometry and will not be analysed in this paper (see [1, 3], for instance).

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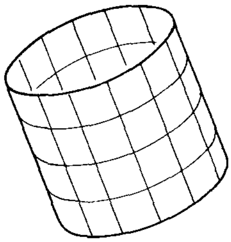

Fig. 1 Cylinder with discretization meshes

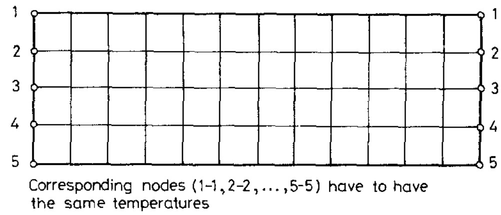

Fig. 2 Developed view of cylinder (corresponding nodes are denoted by the same numbers)

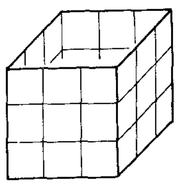

Fig. 3 Box with discretization meshes

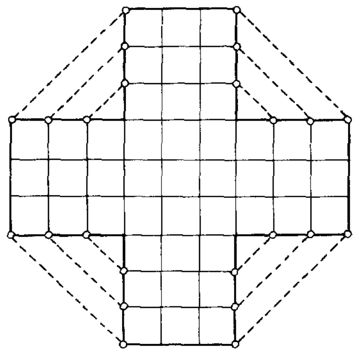

Fig. 4 Developed view of the box (corresponding nodes are joined)

Heat flow in three-dimensional developable structures

Let Ω_D^2 be the developed view of the surface . In order to show the temperature distribution in the region Ω^3, we must join the corresponding nodes at the boundary of Ω_D^2 (see Figs 2 and 4). In this way we obtain the same situation of heat flow as at Ω^3. This may be done by joining corresponding nodes by the elements, or computing the equality of temperatures in corresponding nodes.

Finite element equation

An appropriate mathematical description of the heat conduction process in a material region Ω^3 (three-dimensional) is given by

$$\varrho C_p \frac{\partial \Theta}{\partial t} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial \Theta}{\partial x_j}\right) - Q = 0, \quad i, j = 1, 2, 3 \tag{1}$$

where ϱ is the material density, C_p is the heat capacity, k_{ij} are the elements of the conductivity tensor, Q is the volumetric heat source, t is time, x_i are the spatial coordinates and Θ is temperature. The boundary of the region Ω^3 is defined by $\Gamma^2 = \Gamma^2_\Theta + \Gamma^2_q$ (the boundary is locally two-dimensional), where Γ^2_Θ and Γ^2_q are parts of the boundary for which the temperature and heat flux are specified. The relevant boundary conditions for Eq. (1) may then be expressed by

$$\Theta = \Theta_b \text{ on } \Gamma^2_\Theta \tag{2}$$

and

$$q_i n_i + \left(k_{ij}\frac{\partial \Theta}{\partial x_j}\right) + q_c + q_r = 0 \text{ on } \Gamma^2_q \tag{3}$$

where Θ_b is an applied boundary temperature, q_i is the applied heat flux vector, n_i is the unit outward normal to the boundary Γ_q, q_c is the heat flux due to convection, and q_r is the heat flux due to radiation.

When we describe heat conduction analysis using the development view of the three-dimensional structure into a two-dimensional plane, finite elements are employed to solve in two dimensions. The two-dimensional problem (in region Ω^2) of the transient heat flow is assumed to be described by the equation

$$\varrho C_p \frac{\partial \Theta}{\partial t} - \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial \Theta}{\partial x_j}\right) - Q = 0, \quad i, j = 1, 2 \tag{4}$$

where all notations used are the same as in Eq. (1).

The boundary Γ^2 of the region Ω^2 may be defined by $\Gamma^2 = \Gamma^2_\Theta + \Gamma^2_q$. The boundary conditions Eq. (4) are specified by Eqs (2) and (3). The problem of radiation may also be analysed. In such cases the problems of internal radiation should be discussed. The methods of such analysis will not be undertaken in the present approach.

Assume that within each finite element the temperature field may be approximated by

$$\Theta(x_1, x_2, t) = \sum_{n-1}^{N} \Phi_n(x_1, x_2)\Theta_n(t), \tag{5}$$

or in matrix notation

$$\Theta(x_1, x_2, t) = \Phi^T(x_1, x_2)\Theta(t). \tag{6}$$

In Eq. (5), Φ_n is an N-dimensional vector of interpolation (shape) functions, Θ_n is a vector of nodal point unknowns, and N is the number of nodal points in an element.

An application of the Galerkin method to Eq. (4) produces the following equation

$$\int\limits_{\Omega_e^2} \Phi \left\{ \varrho C_p \Phi^T \frac{\partial \Theta}{\partial t} - \frac{\partial}{\partial x_i}\left(k_{ij} \frac{\partial \Phi^T}{\partial x_j} \Theta \right) - Q \right\} \mathrm{d}\Omega^2 + \\ + \int\limits_{\Gamma_e^2} \Phi \left(q_i n_i + k_{ij} \frac{\partial \Theta^T}{\partial x_j} \Theta n_i + q_c \right) \mathrm{d}\Gamma^2 = 0 \tag{7}$$

Equation (7) may be rewritten using Green's theorem (basically an integration by parts of the second-order derivative term) to give the equation

$$\int\limits_{\Omega_e^2} \varrho C_p \Phi \Phi^T \frac{\partial \Theta}{\partial t} \mathrm{d}\Omega^2 + \int\limits_{\Omega_e^2} \frac{\partial \Phi}{\partial x_i} k_{ij} \frac{\partial \Phi^T}{\partial x_j} \Theta \, \mathrm{d}\Omega^2 = \\ = \int\limits_{\Omega_e^2} \Phi Q \, \mathrm{d}\Omega - \int\limits_{\Gamma_e^2} \Phi (q_i n_i + q_c) \, \mathrm{d}\Gamma^2. \tag{8}$$

Let ϱC_p and k_{ij} be approximated by

$$\varrho C_p = \boldsymbol{\eta}^T \varrho \boldsymbol{C}_p, \\ k_{ij} = \boldsymbol{\eta}^T \boldsymbol{k}_{ij} \tag{9}$$

where $\boldsymbol{\eta}$ is a vector of interpolation functions and $\varrho \boldsymbol{C}_p$ and $\boldsymbol{k}_{ij}$ are vectors of nodal point heat capacities and conductivities, respectively. A similar technique may be used to allow the volumetric heat source within an element to have an arbitrary functional dependence. Thus, let

$$Q = \boldsymbol{\eta}^T \mathbf{Q} \tag{10}$$

where $\mathbf{Q}$ is a vector of nodal point volumetric sources. Substitution of Eqs (9) and (10) into Eq. (8) produces the following equation

$$\int\limits_{\Omega_e^2} \boldsymbol{\eta}^T \varrho \mathbf{C}_p \boldsymbol{\Phi} \boldsymbol{\Phi}^T \mathrm{d}\Omega^2 \frac{\partial \boldsymbol{\Theta}}{\partial t} + \int\limits_{\Omega_e^2} \frac{\partial \boldsymbol{\Phi}}{\partial x_i} \boldsymbol{\eta}^T \mathbf{k}_{ij} \frac{\partial \boldsymbol{\Theta}^T}{\partial x_j} \mathrm{d}\Omega^2 \, \boldsymbol{\Theta} = \\ = \int\limits_{\Omega_e^2} \boldsymbol{\Phi} \boldsymbol{\eta}^T \mathbf{Q} \, \mathrm{d}\Omega^2 - \int\limits_{\Gamma_e^2} \boldsymbol{\Phi} (q_i n_i + q_c) \, \mathrm{d}\Gamma^2. \tag{11}$$

Once the forms of the interpolation functions $\boldsymbol{\Phi}$ and $\boldsymbol{\eta}$ are specified for an element, the integrals in Eq. (11) may be evaluated. Such an evaluation leads to the matrix equation for each element as

$$M\dot{\Theta} + K\Theta = \mathbf{F}_Q + \mathbf{F}$$

where

$$M = \int_{\Omega_e^2} \boldsymbol{\eta}^T \varrho \mathbf{C}_p \boldsymbol{\Phi}\boldsymbol{\Phi}^T \, d\Omega^2,$$

$$K = \int_{\Omega_e^2} \frac{\partial \boldsymbol{\Phi}}{\partial x_i} \boldsymbol{\eta}^T \mathbf{k}_{ij} \frac{\partial \boldsymbol{\Phi}^T}{\partial x_j} \, d\Omega^2,$$

$$\mathbf{F}_Q = \int_{\Omega_e^2} \boldsymbol{\Phi}\boldsymbol{\eta}^T \mathbf{Q} \, d\Omega^2,$$

$$\mathbf{F} = -\int_{\Gamma_e^2} \boldsymbol{\Phi}(q_n + q_c) \, d\Gamma^2.$$

The previous discussion was directed towards the derivation of the equation for a single element. The finite element model for the entire region Ω^3 is obtained through assembly of the element matrices by imposing appropriate interelement continuity requirements on the dependent variable.

Let us consider one thin finite element Ω_e of the area $|\Omega_e|$. The area of the boundary of this element $|\Gamma_e|$ is equal to the area $|\Omega_e|$. Moreover, each element has two boundary surfaces with the same areas.

General remarks. It is important how thin the structure has to be for this method to work. These problems depend on the type of analysis, the kind of structure, the material properties and the boundary conditions, and should be analysed separately for each case.

Example solution

Heat flow in the pipeline element

We will analyse the heat flow in a steel cylinder of wall thickness 0.2 mm and diameter 4 cm. Initial temperature 20°; environmental temperature 20°; properties of steel: thermal conductivity 50 kJ/mKs, specific heat 1000 J/kg K, convection coefficient 0.25 W/m² K. Figure 6 shows a developed view of this cylinder in the situation when the moving heat source occurs on the cylinder. The velocity of the heat source is $v = 0.01$ m/s, and the heat rate is $q = 15 \times 10^4$ W/m. The situation described above may represent the welding process of pipeline elements. In Fig. 6, the heat source is moving through the cutting line. Such a situation certainly has to arise if the heat source is moving around the cylinder. Figure 7 shows a similar example for the velocity 0.01 m/s and heat rate 75,000 W/m. A comparison with the full 3D model is very good. The results are accurate; good agreement is obtained using different meshes (Fig. 5).

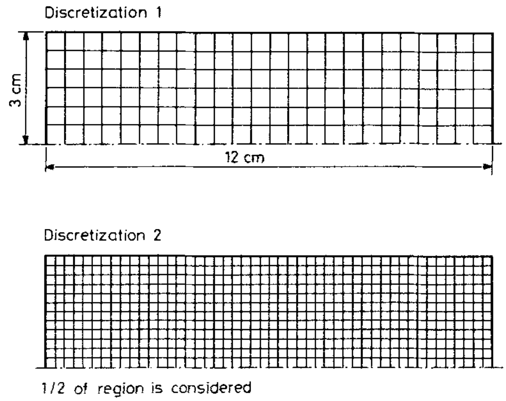

Fig. 5 Discretizations of cylinder

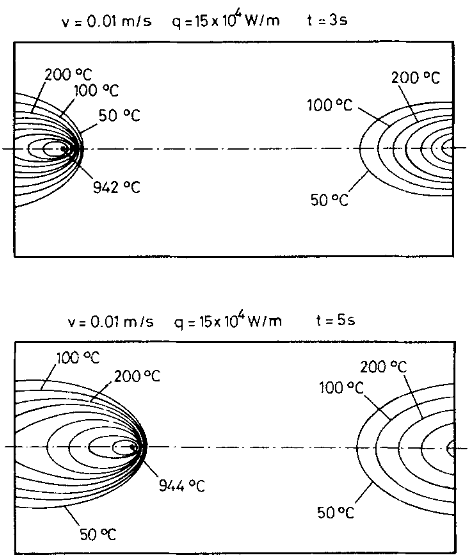

Fig. 6 Heat flow on the developed view of cylinder

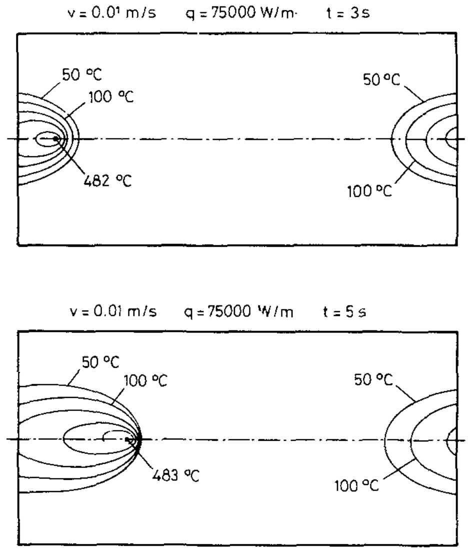

Fig. 7 Heat flow on the developed view of cylinder

Conclusions

The method described in the paper may also be used for more complex problems, such as heat flow on a spherical surface, on joined cylinders, etc. The applications of this method for solving specific technical problems are wide, and will be discussed in the future.

References

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Zusammenfassung — Eine endliche Elemente involvierende Näherung des Problems der thermischen Analyse einer im dreidimensionalen Raum lokalisierten dünnen Struktur wird entwickelt. Das Problem wird auf ein zweidimensionales mit speziellen Randbedingungen reduziert. Die erhaltenen Ergebnisse werden mit der dreidimensionalen Lösung verglichen.

Резюме — Представлен ограниченный элемент подхода к термическому анализу тонкой структуры, определяемой в трехмерном пространстве. Задача решена преобразованием до двумерного пространства со специальными граничными условиями. Обсуждено сопоставление полученных результатов с таковыми при трехмерном решении.